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The Cherenkov radiation in a wave-guide loaded with a moving medium

By R. M. Khan

Department of Mathematics,

City College, Amherst Street, Calcutta, India

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The field of a charge moving along the axis of a cylindrical wave-guide, which is filled with a moving dielectric, is obtained. Simple expressions are derived for the total energy loss due to the Cherenkov radiation and the excitation of plasma waves. The energy of the continuous spectrum is found to be concentrated at certain discrete frequencies, as usual in a waveguide.

1. Introduction

Abele (1952), Akhiezer (1956), Bogdankevich & Bolotovskii (1957), Bouch-Osmolovskii (1963), Lomize & Kurbanov (1961) and many others have studied the waveguide problem in relation to the Cherenkov radiation. In each case the medium within the waveguide is stationary and a point charge or a beam of electrons or a dipole is moving within it. So it is natural that one will be tempted to think of a waveguide loaded with a moving dielectric. Here we consider a dielectric medium moving within an infinite metalic cylinder of radius a and a point charge is also moving along the axis of the cylinder. To write down the fundamental equations we have borrowed some idea of Bolotovskii & Rukhadze (1959) who have discussed certain properties of the field and the energy loss due to the Cherenkov radiation for a moving charge in a moving medium.

Electro-magnetic field intensities have been investigated in section 3; they are consistent with the usual boundary conditions. In section 4 the total energy loss due to the Cherenkov radiation has been derived and as it is usual in waveguide problem the modes of frequencies are obtained in a very simple from. By a little modification it is shown in section 5 that energy loss due to plasma oscillations may be obtained. The energy loss of a charge in a waveguide is generally determined by the retardation force exerted on the charge by the field produced by the charge. We have calculated the total energy loss by the general technique of the Poynting vector. As a consequence the result obtained is not exactly identical with those of others.

2. Phenomenological equations

Let us consider the phenomenological equations of classical electrodynamics for a moving medium which have been used by Ryazanove (1957) and Bolotovski & Rukhade (1959). Let $\epsilon(\omega)$ and μ be the dielectric constant and the permeability of the medium in the rest system and

$$\chi=\epsilon\mu-1 \qquad \ldots(1)$$

If the medium moves with the 4-velocity u_i,

$u_{1,2,3} = \dfrac{u_{x,y,z}}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$, $u_4 = \dfrac{c}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$ (c is the velocity of light in vacuum and "u" is the three dimensional velocity of the medium) then dielectric-magnetic permeability tensor is written in the form

$$\epsilon_{iksl} = \frac{1}{\mu}\left(\delta_{is} + \frac{\chi}{c^2}\, u_i\, u_s\right)\left(\delta_{kl} + \frac{\chi}{c^2}\, u_k u_l\right). \qquad \ldots(2)$$

Maxwell's equations assume the form

$$\left.\begin{aligned} \delta_i F_{ks} + \delta_k F_{si} + \delta_s F_{ik} &= 0 \\ \delta_k H_{ik} &= -\frac{4\pi}{c} j_i \\ H_{ik} &= \epsilon_{iksl}\, F_{sl} \end{aligned}\right\} \qquad \ldots(3)$$

where $\delta_i\left(-\nabla, \frac{1}{c}\,\frac{\partial}{\partial t}\right)$ is the four dimensional gradient. F_{ik} is a field tensor (electric field and magnetic induction field) and H_{ik} is a field tensor (magnetic field and electric induction field).

Let us introduce the 4-potentials A_i in accordance with the relation

$$F_{ik} = \delta_i A_k - \delta_k A_i. \qquad \ldots(4)$$

A supplementary condition about connection of the 4-potentials is

$$\delta_i\left(A_i + \frac{\chi}{c^2}\, u_i\, u_k A_k\right) = 0. \qquad \ldots(5)$$

Under these conditions the equations for potentials are

$$\left[\delta_k{}^2 + \frac{\chi}{c^2}\,(u_k\delta_k)^2\right]\left(\delta_{il} + \frac{\chi}{c^2}\, u_i u_l\right)A_l = -\frac{4\pi}{c}\mu\, j_i. \qquad \ldots(6)$$

These equations are in good agreement with those in a medium at rest ($u = 0$).

In our case we consider the axis of the cylinder as z-axis and the medium is moving with a velocity u in the direction of z-axis. Therefore

$$u_1 = 0 = u_2 \, , \, u_3 = \frac{u}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} \; , \; u_4 = \frac{c}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} \, .$$

Further we assume that a point charge q moves along the z-axis with a velocity v inside the medium which is itself moving with a velocity u.

The components of 4-density vector are

$$\left.\begin{aligned} j_1 = 0 = j_2, \; j_3 &= vq\delta(x)\,\delta(y)\,\delta(z-vt) \\ \text{and } \; j_4 &= -cq\,\delta(x)\,\delta(y)\,\delta(z-vt). \end{aligned}\right\} \quad (7)$$

Taking $A_1 = 0 = A_2$ and the Fourier transform of A_3 and A_4 in the form $A_3 = \int_{-\infty}^{\infty} A_3(\omega)\, e^{i\omega t} d\omega$, $A_4 = \int_{-\infty}^{\infty} A_4(\omega)\, e^{i\omega t} d\omega$, we have from (6) and (7)

$$\left.\begin{aligned} &\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - \frac{\chi}{c^2-u^2}\left(i\omega - u\frac{\partial}{\partial z}\right)^2\right] \times \\ &\quad \left\{\left(1 + \frac{\chi u^2}{c^2-u^2}\right) A_3(\omega) + \frac{\chi c u}{c^2-u^2} A_4(\omega)\right\} = -\frac{2\mu q}{c}\,\delta(x)\delta(y)e^{-i\omega z/v} \\ &\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - \frac{\chi}{c^2-u^2}\left(i\omega - u\frac{\partial}{\partial z}\right)^2\right] \times \\ &\quad \left\{\frac{\chi c u}{c^2-u^2} A_3(\omega) + \left(1 + \frac{\chi c^2}{c^2-u^2}\right) A_4(\omega)\right\} = \frac{2\mu q}{v}\,\delta(x)\delta(y)e^{-i\omega z/v} \end{aligned}\right\} \quad \ldots(8)$$

Putting $A_3(\omega) = -\eta_1(x,y)e^{-i\omega z/v}$, $A_4(\omega) = \eta_2(x,y)e^{-i\omega z/v}$ in (5) and (8) we obtain

$$-\frac{M}{v}\eta_1 + \frac{N}{c}\eta_2 = 0, \quad \ldots(9)$$

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + s^2\right)\eta_1 &= \frac{2\mu q}{cP}\,\delta(x)\,\delta(y) \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + s^2\right)\eta_2 &= \frac{2\mu q}{vQ}\,\delta(x)\,\delta(y) \end{aligned}\right\} \quad \ldots(10)$$

$$\left.\begin{aligned}
&\text{where, } M = 1+\chi\,\frac{u(u+v)}{c^2-u^2}\,,\; N = 1+\frac{\chi}{v}\,\frac{c^2(u+v)}{c^2-u^2}\,,\\
&P = \frac{K}{N}\,,\; Q = \frac{K}{M}\,,\; K = 1+\chi\,\frac{c^2+u^2}{c^2-u^2}\\
&\text{and } s^2 = \frac{\omega^2}{c^2}+\chi\,\frac{\omega^2}{v^2}\,\frac{(u+v)^2}{c^2-u^2}-\frac{\omega^2}{v^2}
\end{aligned}\right\} \qquad \ldots(11)$$

Let,

$$\left.\begin{aligned}
\eta_1 &= L_1 J_0(s\rho)+\frac{\mu q}{2cP}\,N_0(s\rho)\\
\eta_2 &= L_2 J_0(s\rho)+\frac{\mu q}{2vQ}\,N_0(s\rho)
\end{aligned}\right\} \qquad \ldots(12)$$

where, $\rho=\sqrt{x^2+y^2}$.

By (9) and (12) $\quad \dfrac{M}{v}\,L_1-\dfrac{N}{c}\,L_2=0. \qquad \ldots(13)$

Now

$$\left.\begin{aligned}
A_3 &= -\int \eta_1\, e^{i\omega(t-z/v)}\,d\omega\\
A_4 &= \int \eta_2\, e^{i\omega(t-z/v)}\,d\omega
\end{aligned}\right\} \qquad \ldots(14)$$

3. Field components

By the help of (3), (4) and (14)

$$\left.\begin{aligned}
E_x(\omega) &= \frac{sx}{\rho}\left[L_2 J_1(s\rho)+\frac{\mu q}{2vQ}\,N_1(s\rho)\right] e^{i\omega(t-z/v)}\\
E_y(\omega) &= \frac{sy}{\rho}\left[L_2 J_1(s\rho)+\frac{\mu q}{2vQ}\,N_1(s\rho)\right] e^{i\omega(t-z/v)}\\
E_z(\omega) &= i\omega\left(\frac{\eta_1}{c}-\frac{\eta_2}{v}\right) e^{i\omega(t-z/v)}\\
H_x(\omega) &= -\frac{sy}{\mu\rho}\left[\left(1+\chi\frac{u^2}{c^2-u^2}\right)\left\{L_1 J_1(s\rho)+\frac{\mu q}{2cP}\,N_1(s\rho)\right\}\right.\\
&\qquad\left.-\frac{\chi cu}{c^2-u^2}\left\{L_2 J_1(s\rho)+\frac{\mu q}{2vQ}\,N_1(s\rho)\right\}\right] e^{i\omega(t-z/v)}\\
H_y(\omega) &= \frac{sx}{\mu\rho}\left[\left(1+\chi\frac{u^2}{c^2-u^2}\right)\left\{L_1 J_1(s\rho)+\frac{\mu q}{2cP}\,N_1(s\rho)\right\}\right.\\
&\qquad\left.-\frac{u\chi c}{c^2-u^2}\left\{L_2 J_1(s\rho)+\frac{\mu q}{2vQ}\,N_1(s\rho)\right\}\right] e^{i\omega(t-z/v)}\\
H_z(\omega) &= 0,
\end{aligned}\right\} \qquad \ldots(15)$$

In cylindrical co-ordinates ($\rho, \theta, z,$)

$$\left.\begin{aligned} E_\rho(\omega) &= s\left[L_2 J_1\,(s\rho) + \frac{\mu q}{2vQ}\, N_1(s\rho) \right] e^{i\omega(t-z/v)} \\ H_\theta(\omega) &= \frac{s}{\mu}\left[\left(1+\chi\frac{u^2}{c^2-u^2}\right)\left\{ L_1 J_1\,(s\rho) + \frac{\mu q}{2cQ}\, N_1\,(s\rho) \right\} \right. \\ &\quad \left. + \chi\frac{cu}{c^2-u^2}\left\{ L_2 J_1\,(s\rho) + \frac{\mu q}{2vQ}\, N_1(s\rho) \right\} \right] e^{i\omega(t-z/v)} \end{aligned}\right\} \qquad \ldots(16)$$

On the surface of the cylinder (*i.e.* $\rho=a$) $E_z=0$.
For this

$$\left(\frac{L_1}{c} - \frac{L_2}{v} \right) J_o\,(sa) + \frac{\mu q}{2}\left(\frac{1}{c^2P} + \frac{1}{v^2Q} \right) N_o(sa)=0. \qquad \ldots(17)$$

By (13) and (17)

$$\left.\begin{aligned} L_1 &= -\frac{\mu q}{2c}\,\frac{N}{K}\,\frac{N_o(sa)}{J_o(sa)} \\ L_2 &= -\frac{\mu q}{2v}\,\frac{M}{K}\,\frac{N_o(sa)}{J_o(sa)} \end{aligned}\right\}. \qquad \ldots(18)$$

Now $Re\,E_\rho = -\dfrac{q}{v}\displaystyle\int_o^\infty \frac{\mu s M}{K}\left\{ \frac{N_o(sa)}{J_o(sa)}\, J_1(s\rho) - N_1(s\rho) \right\} \cos\omega\,(t-z/v) d\omega$

$$Re\,H_\theta = -\frac{q}{c}\int_o^\infty s\left\{ \frac{N_o(sa)}{J_o(sa)}\, J_1\,(s)\rho - N_1\,(s\rho) \right\} \cos\omega\,(t-z/v) d\omega$$

$$E_z = -iq\int_o^\infty \frac{\mu\,\omega}{K}\left(\frac{N}{c^2} - \frac{M}{v^2} \right)\left\{ \frac{N_o(sa)}{J_o(sa)}\, J_o\,(s\rho) - N_o\,(s\rho) \right\} e^{i\omega(t-z/v)}. \qquad (19)$$

4. The radiation energy loss

Total energy radiated by the particle inside the cylinder per unit time

is $\dfrac{dW_1}{dt} = \dfrac{c}{4\pi}\displaystyle\int_{z=-\infty}^{\infty}\int_{\rho=0}^{a} (\,Re\,E_\rho . Re\,H_\theta\,)\; 2\pi d\rho dz.$

Taking the values of ReE_ρ and ReH_θ from (19) and using the formula

$$\int_{-\infty}^{\infty} \cos\omega\,(t-z/v)\,\cos\omega\,(t-z/v)\,dz = \pi\,v\,\delta\,(\,\omega-\omega'\,)$$

we have $\frac{dW_1}{dt} = \frac{\pi q^2}{2} \int_{\omega=0}^{\infty} \int_{\rho=0}^{a} \mu \ s^2 \ \frac{M}{K} \left\{ \frac{N_0(sa)}{J_0(sa)} J_1(s\rho) - N_1(s\rho) \right\}^2 \rho d\rho d\omega$

$$= \frac{q^2}{\pi} \int_{\omega=0}^{\infty} \frac{1+\chi \frac{u(u+v)}{c^2-u^2}}{1+\chi \frac{c^2+u^2}{c^2-u^2}} \cdot \frac{1}{J_0^2(sa)} d\omega \quad . \qquad \ldots(20)$$

This integral can be determined by the residues at the poles of the expression under the integrand. Because of the poles the integral becomes a series and the continuous spectrum is replaced by a discrete spectrum characteristic of a waveguide.

Poles are obtained at the zeroes of $J_0(sa)=0$. If $J_0(sa)=0$ for $\omega=\omega_k$ and we write $sa=\alpha_k$ for this value of ω, then,

$$\frac{dW_1}{dt} = \frac{2q^2}{a^2} \sum \mu(\omega_k) \frac{1+\chi(\omega_k) \frac{u(u+v)}{c^2-u^2}}{1+\chi(\omega_k) \frac{c^2+u^2}{c^2-u^2}} \cdot \frac{1}{J_1^2(\alpha_k)} \left\{ \frac{1}{\frac{ds^2}{d\omega}} - \frac{\frac{d^2 s}{d\omega^2}}{2 \frac{ds}{d\omega}} \right\}_{\omega=\omega_k} . \qquad \ldots(21)$$

The summation is taken over all harmonics for which the radiation condition $s^2>0$ is satisfied.

If the medium is stationary then $u=0$. In this case the total energy loss by the particle within the waveguide per unit time is

$$\frac{dW_2}{dt} = \frac{2q^2}{a^2} \sum_k \frac{1}{\epsilon(\omega'_k)} - \frac{1}{J_1^2(\alpha'_k)} \left\{ \frac{1}{\frac{ds'^2}{d\omega}} - \frac{\frac{d^2 s'}{d\omega^2}}{2 \frac{ds'}{d\omega}} \right\}_{\omega=\omega_k} \qquad \ldots(22)$$

where $s'^2 = \frac{\omega^2}{v^2} (\epsilon\mu\beta^2-1)$, $\beta = \frac{v}{c}$, $\alpha'_k = (s'a)_{\omega=\omega'_k}$ and $J_0(s'a)=0$ for $\omega=\omega'_k$.

For dispersionless medium ϵ and μ are independent of frequency and then from (22)

$$\frac{dW_2}{dt} = \frac{2q^2}{a^2 \epsilon} \sum_k \frac{1}{J_1^2(\alpha''_k)} \left(\frac{1}{\frac{ds''^2}{d\omega}} \right)_{\omega=\omega_k''} \qquad \ldots(23)$$

where $s''^2 = \frac{\omega^2}{v^2} (\epsilon\mu\beta^2-1)$, $\alpha''=(s''a)_{\omega=\omega''_k}$ and $J_0(s''a)=0$ for $\omega=\omega''_k$.

The summations of (22) and (23) are restricted by the conditions $s'^2>0$ and $s''^2>0$.

Following Bolotoviskii (1959) and others the energy loss of the charge per unit length of the path is

$$\frac{dW}{dz}=q.E_z\Big|_{\rho\to 0}^{z\to vt}=-q.^2Re\int_0^{\infty}\frac{\mu}{K}\left(\frac{N}{c^2}-\frac{M}{v^2}\right)\frac{N_0(sa)}{J_0(sa)}i\omega d\omega \text{ by (19).}$$

For stationary medium $u=0$ and

$$\frac{dW}{dz}=-q.^2Re\int_0^{\infty}\frac{is'^2}{\epsilon\,\omega}\frac{N_0(s'a)}{J_0(s'a)}\,d\omega=-\frac{2q^2}{a^2}\sum_k\left[\frac{s'}{\epsilon\,\omega}\frac{1}{J_1^2(s'a)}\frac{1}{\frac{ds'}{d\omega}}\right]_{\omega=\omega'_k}$$

when $s'^2>0$ and $J_0(s'a)=0$ for $\omega=\omega'_k$.

For dispersionless medium, $\dfrac{dW}{dz}=-\dfrac{2q^2}{\epsilon a^2}\sum_k\dfrac{1}{J_1^2(\alpha''_k)}$. It is identical with Akhiezer's result (1956).

5. Energy loss due to excitation of plasma waves :

If the medium moves with a large velocity $(1-\frac{u^2}{c^2}\ll 1)$ then we may put $\mu=1$, $\epsilon=1-\frac{\omega_0^2}{\omega^2}$. In this case the propagation relation is the same as for an electron plasma (Bolotovskii & Rukhadze 1959) and the equations (10) convert to

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\alpha^2\right)\eta_1&=\frac{2q}{cP'}\delta(x)\delta(y)\\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\alpha^2\right)\eta_2&=\frac{2q}{vQ'}\delta(x)(\delta y)\end{aligned}\right\}\quad\ldots(24)$$

where $P'=\dfrac{K'}{N'}$, $Q'=\dfrac{K'}{M'}$, $K'=1-\dfrac{\omega_0^2}{\omega^2}\dfrac{c^2+u^2}{c^2-u^2}$, $M'=1-\dfrac{\omega_0^2}{\omega^2}\dfrac{u(u+v)}{c^2-u^2}$,

$$N'=1-\frac{\omega_0^2}{\omega^2}\frac{c^2(u+v)}{v(c^2-u^2)},\quad \alpha^2=\omega^2\left[\frac{1}{v^2}+\frac{\omega_0^2}{\omega^2}\frac{(u+v)^2}{v^2(c^2-u^2)}-\frac{1}{c^2}\right].$$

Taking $\eta_1=L_3\,I_0(\alpha\rho)-\dfrac{q}{\pi cP'}K_0(\alpha\rho)$, $\eta_2=L_4I_0(\alpha\rho)-\dfrac{q}{\pi vQ'}K_0(\alpha\rho)$

and proceeding as section 2 and 3 we have

$$\left.\begin{aligned} ReE_\rho &= -\frac{2\omega}{\pi v}\int_0^\infty \frac{\alpha M'}{K'}\left[\frac{K_0(\alpha a)}{I_0(\alpha a)}I_1(\alpha\rho)+K_1(\alpha\rho)\right]\cos\omega\left(t-\frac{z}{v}\right)d\omega \\ ReH_\theta &= -\frac{2q}{\pi c}\int_0^\infty \alpha\left[\frac{K_0(\alpha a)}{I_0(\alpha a)}I_1(\alpha\rho)+K_1(\alpha\rho)\right]\cos\omega\left(t-\frac{z}{v}\right)d\omega \end{aligned}\right\} \quad (25)$$

Energy loss per unit time is

$$\frac{dW_3}{dt}=\frac{c}{4\pi}\int_{z=-\infty}^{\infty}\int_{\rho=0}^{a}(ReE\rho\ ReH_\theta)\,2\pi\rho d\rho dz$$

$$=\frac{q^2}{\pi}\int_{\omega=0}^{\alpha}\frac{\omega^2(c^2-u^2)-\omega_0^2\,u\,(u+v)}{\omega^2(c^2-u^2)-\omega_0^2(c^2+u^2)}\cdot\frac{1}{I_0^2(\alpha a)}\,d\omega. \quad \ldots(26)$$

The integration can be determined by residue method and the poles are obtained from $\omega^2(c^2-u^2)-\omega_0{}^2(c^2+u^2)=0$

since $\omega>0$, $\omega_k=\omega_0\sqrt{\left(\frac{c^2+u^2}{c^2-u^2}\right)}$.

$$\text{Thus, } \frac{dW_3}{dt}=q^2\frac{\sqrt{(c^4-u^4)}-u\,(u+v)}{\sqrt{(c^4-u^4)}+c^2+u^2}\cdot\frac{1}{I_0^2\left(\frac{a\omega_0}{cv}\sqrt{\frac{c^4-u^2v^2}{c^2-u^2}}\right)}. \quad \ldots(27)$$

This investigation may provide some information about the nature of the electron plasma as the state of ionization.

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